Replica symmetric solutions of the graph-bipartitioning problem with fixed, finite valence

1988 J. Phys. A: Math. Gen. 21 427

(http://iopscience.iop.org/0305-4470/21/2/022)

J. Phys. A: Math. Gen. **21** (1988) 427-440. Printed in the UK

Replica symmetric solutions of the graph-bipartitioning problem with fixed, finite valence

Wuwell Liao

Physics Department, Princeton University, Princeton, NJ 08544, USA

Received 22 June 1987, in final form 2 September 1987

Abstract. Using the replica method, we present the replica symmetric solutions of the graph-bipartitioning problem with fixed, finite valence. With the constraint $\Sigma_{i=1}^{N} S_i = 0$ strictly enforced, we are able to find another solution which gives a lower cost function than that given by the spin glass solution.

1. Introduction

Recently, methods of the statistical mechanics of random systems have been frequently applied to the study of hard optimisation problems. Several authors have discussed the application of the replica method [1] in the combinatorial optimisation problems [2, 3]. The central idea is to map the optimisation problem onto a certain spin Hamiltonian. Then the quantity to be optimised, hereafter referred to as the cost function, is related to the ground-state energy of the corresponding spin Hamiltonian. In general, the major hurdle is the computation of the quenched average. However, there are usually additional constraints imposed on the spin variables. For example, in the study of the travelling salesman problem, we have to either introduce permutation group elements [4] or use p-vector spin variables [5] and take the limit $p \to 0$ at the end. In the so-called graph-bipartitioning problem [2], we have the additional constraint of $\Sigma_{i=1}^{N} S_i = 0$. In treating these constraints, great care must be exercised in order to find the optimal solution.

In a recent paper [6], I demonstrated that, in the graph-bipartitioning problem with average, finite valence [7-9], it is incorrect to replace the constraint, $\Sigma_{i=1}^{N} S_i = 0$, by a soft version of the constraint, namely $\exp[-\lambda(\Sigma_{i=1}^{N} S_i)^2]$ with $\lambda \gg 1$, in the calculation of the canonical partition function. It was explicitly shown that the constraint must be strictly enforced in order to obtain the optimal solution consistent with the exact result on the size of the infinite cluster by Erdos and Renyi [10].

In this paper, the same technique is applied to the graph-bipartitioning problem with fixed, finite valence. In many aspects, this approach is quite similar to that of Wong and Sherrington [11]. However, with the constraint strictly enforced, we are able to find another solution which gives a lower cost function than that given by the spin glass solution. Unfortunately, the known numerical study [12] was done with a soft version of the constraint (i.e. with the exponential penalty term, $\exp[-\lambda(\Sigma_{i=1}^{N} S_i)^2]$, $\lambda \gg 1$, in the partition function). We are thus unable to confront our solution with 'experimental' data.

The organisation of the paper is as follows: In § 2 we define the model and show details of the calculation. The replica method is used to obtain the replica symmetric solutions. In § 3, some suggestions for future work are discussed.

2. The model and the replica theory

The problem that we consider is as follows. We are given a set of vertices $V = (V_1, V_2, \ldots, V_N)$, with N even, and a set of edges $E = \{(V_i, V_j); i \neq j\}$. Let each vertex be connected with b edges, where $b \in \{1, 2, \ldots, N-1\}$. The bipartitioning problem is to divide V into two parts of equal size in such a way as to minimise the number of edges, N_c, connecting these two parts. N_c is thus our cost function. We are then interested in the behaviour of N_c/N, in the limit $N \to \infty$, as a function of b.

Following Fu and Anderson [2], we attach a spin variable $S_i = \{\pm 1\}$ to each vertex V_i. We want to divide V into two equal parts, say G_1 and G_2. Then $S_i = 1$ means that the vertex V_i belongs to G_1; $S_i = -1$ means that the vertex V_i belongs to G_2. We also use the bond variable, J_{ij}, to represent the edge (V_i, V_j). Then the condition of fixed, finite valence is equivalent to

$$J_{ij} \in \{0, J\} \qquad \text{with} \qquad J_{ii} = 0 \tag{2.1}$$

$$\sum_{j=1}^{N} J_{ij} = bJ \qquad \text{for all } i = 1, 2, \ldots, N \tag{2.2}$$

$J_{ij} = J$ would correspond to the case where the edge (V_i, V_j) is present; $J_{ij} = 0$ would correspond to the case where the edge (V_i, V_j) is absent. Then the cost function, averaged over all possible bond configurations, is:

$$\langle N_c \rangle_{av} = \left\langle \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{J_{ij}}{J} \frac{(1 - S_i S_j)}{2} \right\rangle_{av} = \frac{Nb}{4} + \frac{1}{2J} \langle H \rangle_{av} \tag{2.3}$$

$$H = -\frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} J_{ij} S_i S_j \qquad \text{with} \qquad \sum_{i=1}^{N} S_i = 0. \tag{2.4}$$

The constraint, $\Sigma_{i=1}^{N} S_i = 0$, guarantees that we indeed divide V into two parts of equal size. Here $\langle \ldots \rangle_{av}$ means the average over all possible bond configurations. Hence, to minimise $\langle N_c \rangle_{av}$ is to find the ground-state energy of the Hamiltonian in equation (2.4), subject to the corresponding constraint.

Using the replica method, we now proceed to compute the quenched free energy:

$$\langle \ln Z \rangle_{av} = \lim_{m \to 0} \frac{\langle Z^m \rangle_{av} - 1}{m} \tag{2.5}$$

$$Z = \operatorname*{Tr}_{\Sigma_{i=1}^{N} S_i = 0} \exp\left(\frac{\beta}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} J_{ij} S_i S_j \right) \tag{2.6}$$

$$\begin{aligned} \langle Z^m \rangle_{av} &= \frac{1}{g} \operatorname*{Tr}_{\{J_{ij} = 0, J\}} \prod_{i=1}^{N} \delta\left(\sum_{j=1}^{N} J_{ij} - bJ \right) \mathrm{Tr}' \exp\left(\frac{\beta}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} J_{ij} \sum_{\alpha=1}^{m} S_i^\alpha S_j^\alpha \right) \\ &= \frac{1}{g} \int_0^{2\pi} \left(\prod_{i=1}^{N} \frac{\mathrm{d}y_i}{2\pi} \right) \exp\left(-\mathrm{i} \sum_{i=1}^{N} b y_i \right) \\ &\quad \times \mathrm{Tr}' \prod_{i<j}^{N} \left[1 + \exp\left(\beta J \sum_{\alpha=1}^{m} S_i^\alpha S_j^\alpha + \mathrm{i} y_i + \mathrm{i} y_j \right) \right] \end{aligned} \tag{2.7}$$

$$\mathrm{Tr}' \equiv \operatorname*{Tr}_{\Sigma_{i=1}^{N} S_i^{\alpha_m} = 0} \cdots \operatorname*{Tr}_{\Sigma_{i=1}^{N} S_i^{\alpha_1} = 0} \tag{2.8}$$

where we have used the integral representation of the Kronecker δ function. We also have:

$$g=\operatorname*{Tr}_{\{J_{ij}=0,J\}}\prod_{i=1}^{N}\delta\left(\sum_{j=1}^{N}J_{ij}-bJ\right)$$
$$=\int_0^{2\pi}\left(\prod_{i=1}^{N}\frac{\mathrm{d}y_i}{2\pi}\right)\exp\left(-\mathrm{i}\sum_{i=1}^{N}by_i\right)\prod_{i<j}^{N}\{1+\exp[\mathrm{i}(y_i+y_j)]\}. \tag{2.9}$$

Now we notice that, if we replace:

$$\prod_{i<j}^{N}\{1+\exp[\mathrm{i}(y_i+y_j)]\} \tag{2.10}$$

by

$$\prod_{i<j}^{N}\left(1+\exp[\mathrm{i}(y_i+y_j)]+\sum_{k=b+1}^{\infty}C_k\exp[\mathrm{i}k(y_i+y_j)]\right) \tag{2.11}$$

we do not affect the value of g, provided that the new series, in equation (2.11), is convergent. This is the motivation for introducing:

$$\exp\left(\sum_{a=1}^{b}\frac{(-1)^{a+1}}{a}Z^a\right)=1+Z+\frac{(-1)^{b+1}}{b+1}Z^{b+1}+\sum_{k=b+2}^{\infty}C_kZ^k. \tag{2.12}$$

Equation (2.12) is easily proved, using the mathematical induction on b. Hence, we arrive at:

$$g=\int_0^{2\pi}\left(\prod_{i=1}^{N}\frac{\mathrm{d}y_i}{2\pi}\right)\exp\left(-\mathrm{i}\sum_{i=1}^{N}by_i\right)\prod_{i<j}^{N}\exp\left(\sum_{a=1}^{b}\frac{(-1)^{a+1}}{a}\exp[\mathrm{i}a(y_i+y_j)]\right)$$
$$=\int_0^{2\pi}\left(\prod_{i=1}^{N}\frac{\mathrm{d}y_i}{2\pi}\right)\exp\left(-\mathrm{i}\sum_{i=1}^{N}by_i\right)\prod_{a=1}^{b}\exp\left(-\frac{(-1)^{a+1}}{2a}\sum_{i=1}^{N}\exp(2\mathrm{i}ay_i)\right)$$
$$\times\exp\left[\frac{(-1)^{a+1}}{2a}\left(\sum_{i=1}^{N}\exp(\mathrm{i}ay_i)\right)^2\right]. \tag{2.13}$$

Using the Gaussian transform:

$$\exp(\lambda x^2)=\int_{-\infty}^{\infty}\left(\frac{N}{2\pi}\right)^{1/2}\mathrm{d}r\exp[-\tfrac{1}{2}Nr^2+(2\lambda N)^{1/2}xr] \tag{2.14}$$

we get:

$$g=\int_{-\infty}^{\infty}\left[\prod_{a=1}^{b}\left(\frac{N}{2\pi}\right)^{1/2}\mathrm{d}r_a\right]\exp\left(-\tfrac{1}{2}N\sum_{a=1}^{b}r_a^2+N\ln X\right) \tag{2.15}$$

$$X=\int_0^{2\pi}\frac{\mathrm{d}y}{2\pi}\exp\left\{-\mathrm{i}by+\sum_{a=1}^{b}\left[\left(\frac{(-1)^{a-1}}{a}N\right)^{1/2}r_a\exp(\mathrm{i}ay)-\frac{(-1)^{a+1}}{2a}\exp(2\mathrm{i}ay)\right]\right\}. \tag{2.16}$$

In the limit $N\to\infty$, we only have to find out the leading term in N inside the curly brace of equation (2.16). We first notice that:

$$\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^b\mathrm{e}^{f(z)}=\mathrm{e}^f(f')^b+\frac{b(b-1)}{2}\mathrm{e}^f(f')^{b-2}(f'')+\dots. \tag{2.17}$$

Hence, the leading term in N, after the integration over y, is $(\sqrt{N}r_1)^b/b!$, and the next term is of the order of $(\sqrt{N})^{b-1}$, and so on. By the saddle-point approximation, we finally get:

$$\lim_{N\to\infty} g \approx \exp\left[N\left(\frac{b}{2}[\ln(Nb)-1]-\ln(b!)\right)\right]. \tag{2.18}$$

Following the derivation for g, equation (2.7) can be written as:

$$\langle Z^m\rangle_{\rm av} = \frac{1}{g}\int_0^{2\pi}\left(\prod_{i=1}^N \frac{\mathrm{d}y_i}{2\pi}\right)\exp\left(-\mathrm{i}\sum_{i=1}^N by_i\right)\mathrm{Tr}'$$
$$\times\prod_{a=1}^b \exp\left[\frac{1}{2}\sum_{i=1}^N\sum_{j=1}^N \frac{(-1)^{a+1}}{a}\exp\left(a\beta J\sum_{\alpha=1}^m S_i^\alpha S_j^\alpha + \mathrm{i}a(y_i+y_j)\right)\right]. \tag{2.19}$$

Following the derivations of equations (2.15), (2.17) and (2.18), it is easily proved that in the thermodynamic limit only the term with $a=1$ in equation (2.19) matters. Using equation (2.14) and:

$$\exp\left(\beta J\sum_{\alpha=1}^m S_i^\alpha S_j^\alpha\right) = A_0 + \sum_{l=1}^\infty A_l \sum_{\alpha_1<\ldots<\alpha_l}^m (S_i^{\alpha_1}\ldots S_i^{\alpha_l})(S_j^{\alpha_1}\ldots S_j^{\alpha_l}) \tag{2.20}$$

$$A_0=\cosh^m(\beta J) \qquad A_l=\tanh^l(\beta J)\cosh^m(\beta J) \tag{2.21}$$

we derive the following expression (letting $r_1=r'$):

$$\langle Z^m\rangle_{\rm av} = \frac{1}{g}\int_{-\infty}^\infty\left(\frac{N}{2\pi}\right)^{1/2}\mathrm{d}r'\left[\prod_{l=1}^\infty\prod_{\alpha_1<\ldots<\alpha_l}^m\left(\frac{N}{2\pi}\right)^{1/2}\mathrm{d}q_{\alpha_1\ldots\alpha_l}\right]$$
$$\times\exp\left[-\frac{N}{2}\left(r'^2+\sum_{l=1}^\infty\sum_{\alpha_1<\ldots<\alpha_l}^m q^2_{\alpha_1\ldots\alpha_l}\right)\right]$$
$$\times\int_0^{2\pi}\left(\prod_{i=1}^N\frac{\mathrm{d}y_i}{2\pi}\right)\exp\left(-\sum_{i=1}^N[\mathrm{i}by_i+\tfrac{1}{2}\exp(\beta Jm+2\mathrm{i}y_i)-(NA_0)^{1/2}r'\exp(\mathrm{i}y_i)]\right)$$
$$\times\mathrm{Tr}'\exp\left[\sum_{l=1}^\infty (NA_l)^{1/2}\sum_{\alpha_1<\ldots<\alpha_l}^m q_{\alpha_1\ldots\alpha_l}\left(\sum_{i=1}^N S_i^{\alpha_1}\ldots S_i^{\alpha_l}\exp(\mathrm{i}y_i)\right)\right]. \tag{2.22}$$

Again, using the integral representation of the Kronecker δ function, we find that:

$$\mathrm{Tr}'\exp\left[\sum_{l=1}^\infty (NA_l)^{1/2}\sum_{\alpha_1<\ldots<\alpha_l}^m q_{\alpha_1\ldots\alpha_l}\left(\sum_{i=1}^N S_i^{\alpha_1}\ldots S_i^{\alpha_l}\exp(\mathrm{i}y_i)\right)\right]$$
$$=\int_0^{2\pi}\prod_{\alpha=1}^m\frac{\mathrm{d}X_\alpha}{2\pi}\left[\mathop{\mathrm{Tr}}_{\{S^\alpha\}}\exp\left(\mathrm{i}\sum_{\alpha=1}^m X_\alpha S^\alpha+\sum_{l=1}^\infty (NA_l)^{1/2}\right.\right.$$
$$\left.\left.\times\sum_{\alpha_1<\ldots<\alpha_l}^m q_{\alpha_1\ldots\alpha_l}S^{\alpha_1}\ldots S^{\alpha_l}\exp(\mathrm{i}y)\right)\right]^N. \tag{2.23}$$

Hence, for large N, we get the following expression for $\langle Z^m\rangle_{\rm av}$:

$$\langle Z^m\rangle_{\rm av}\approx\frac{1}{g}\int_{-\infty}^\infty\left(\frac{N}{2\pi}\right)^{1/2}\mathrm{d}r'\left[\prod_{l=1}^\infty\prod_{\alpha_1<\ldots<\alpha_l}^m\left(\frac{N}{2\pi}\right)^{1/2}\mathrm{d}q_{\alpha_1\ldots\alpha_l}\right]$$
$$\times\exp\left[-N\left(\tfrac{1}{2}r'^2+\tfrac{1}{2}\sum_{l=1}^\infty\sum_{\alpha_1<\ldots<\alpha_l}^m q^2_{\alpha_1\ldots\alpha_l}-\frac{1}{N}\ln(\mathrm{int})\right)\right] \tag{2.24}$$

$$\text{int} = \int_0^{2\pi} \prod_{\alpha=1}^{m} \frac{dX_\alpha}{2\pi} \Bigg[\operatorname*{Tr}_{\{S^\alpha\}} \exp\Big(i \sum_{\alpha=1}^{m} X_\alpha S^\alpha \Big)$$

$$\times \Big((NA_0)^{1/2} r' + \sum_{l=1}^{\infty} (NA_l)^{1/2} \sum_{\alpha_1 < \ldots < \alpha_l}^{m} q_{\alpha_1 \ldots \alpha_l} S^{\alpha_1} \ldots S^{\alpha_l} \Big)^b \frac{1}{b!} \Bigg]^N . \tag{2.25}$$

Let us now make the replica symmetry assumption and let:

$$q_{\alpha_1 \ldots \alpha_l} = (bA_l)^{1/2} Q_l \qquad r' = (bA_0)^{1/2} r. \tag{2.26}$$

The saddle-point approximation then gives:

$$\langle Z^m \rangle_{av} \approx \frac{1}{g} \exp\Bigg\{ -N \Bigg[\frac{bA_0}{2} r^2 + \frac{b}{2} \sum_{l=1}^{\infty} A_l Q_l^2 \binom{m}{l} - \frac{1}{N} \ln(\text{int}) \Bigg] \Bigg\} \tag{2.27}$$

$$\text{int} = \int_0^{2\pi} \prod_{\alpha=1}^{m} \frac{dX_\alpha}{2\pi} \Bigg[\operatorname*{Tr}_{\{S^\alpha\}} \exp\Big(i \sum_{\alpha=1}^{m} X_\alpha S^\alpha \Big)$$

$$\Big((Nb)^{1/2} A_0 r + \sum_{l=1}^{\infty} (Nb)^{1/2} A_l Q_l \sum_{\alpha_1 < \ldots < \alpha_l}^{m} S^{\alpha_1} \ldots S^{\alpha_l} \Big)^b \frac{1}{b!} \Bigg]^N . \tag{2.28}$$

We also have to look for the saddle points of:

$$\lim_{m \to 0} \Bigg[\frac{bA_0}{2} r^2 + \frac{b}{2} \sum_{l=1}^{\infty} A_l Q_l^2 \binom{m}{l} - \frac{1}{N} \ln(\text{int}) \Bigg]. \tag{2.29}$$

Let us now introduce the auxiliary field distribution function $\pi(h)$:

$$Q_l \equiv \int_{-\infty}^{\infty} \tanh^l(\beta h) \pi(h) \, dh \tag{2.30}$$

$$\int_{-\infty}^{\infty} \pi(h) \, dh = 1. \tag{2.31}$$

The self-consistency check for equations (2.30) and (2.31) will be presented in the appendix. Putting equation (2.30) into equation (2.27), we get

$$\langle Z^m \rangle_{av} \approx \frac{1}{g} \exp\Bigg[-N \Bigg(\frac{bA_0}{2} r^2 - \frac{b}{2} \cosh^m(\beta J) - \frac{1}{N} \ln(\text{int})$$

$$+ \frac{b}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \pi(h) \pi(h') \, dh \, dh' \frac{\cosh^m(\beta J)}{\cosh^m(\beta z)} e^{\beta z m} \Bigg) \Bigg] \tag{2.32}$$

$$\text{int} \equiv \left(\frac{(Nb)^{b/2}}{b!} \right)^N \int_0^{2\pi} \left(\prod_{\alpha=1}^{m} \frac{dX_\alpha}{2\pi} \right) \Bigg\{ \operatorname*{Tr}_{\{S^\alpha\}} \exp\Big(i \sum_{\alpha=1}^{m} X_\alpha S^\alpha \Big)$$

$$\times \Bigg[(r-1) \cosh^m(\beta J) + \int_{-\infty}^{\infty} \pi(h) \, dh \frac{\cosh^m(\beta J)}{\cosh^m(\beta y)} \exp\Big(\beta y \sum_{\alpha=1}^{m} S^\alpha \Big) \Bigg]^b \Bigg\}^N \tag{2.33}$$

$$\beta y = \tanh^{-1}[\tanh(\beta J) \tanh(\beta h)] \tag{2.34}$$

$$\beta z = \tanh^{-1}[\tanh(\beta J) \tanh(\beta h) \tanh(\beta h')]. \tag{2.35}$$

We first notice that:

$$\left\{\operatorname*{Tr}_{\{S^\alpha\}} \exp\left(\mathrm{i} \sum_{\alpha=1}^{m} X_\alpha S^\alpha\right)\left[(r-1)\cosh^m(\beta J) + \int_{-\infty}^{\infty} \pi(h)\,\mathrm{d}h \frac{\cosh^m(\beta J)}{\cosh^m(\beta y)} \exp\left(\beta y \sum_{\alpha=1}^{m} S^\alpha\right)\right]^b\right\}^N$$
$$= \sum \prod_{i=1}^{N} \left[\binom{b}{k_i}(r-1)^{k_i} \cosh^{mk_i}(\beta J) \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{a_i=1}^{b-k_i} \times\left(\pi(h_{a_i})\,\mathrm{d}h_{a_i} \frac{\cosh^m(\beta J)}{\cosh^m(\beta y_{a_i})}\right)\right] \cosh^m\left(\beta \sum_{a_i=1}^{b-k_i} y_{a_i}\right)$$
$$\times \prod_{i=1}^{N} \prod_{\alpha=1}^{m} \left\{\mathrm{e}^{\mathrm{i}X_\alpha}\left[1+\tanh\left(\beta \sum_{a_i=1}^{b-k_i} y_{a_i}\right)\right] + \mathrm{e}^{-\mathrm{i}X_\alpha}\left[1-\tanh\left(\beta \sum_{a_i=1}^{b-k_i} y_{a_i}\right)\right]\right\} \tag{2.36}$$

where Σ denotes the sum over all possible sets of $\{k_i;\ i=1,2,\ldots,N\}$ with $k_i \in \{0,1,2,\ldots,b\}$. Let us look at a particular set of $\{k_i;\ i=1,2,\ldots,N\}$ in Σ. We get:

$$\int_0^{2\pi} \prod_{\alpha=1}^{m} \frac{\mathrm{d}X_\alpha}{2\pi} \prod_{i=1}^{N} \left\{\mathrm{e}^{\mathrm{i}X_\alpha}\left[1+\tanh\left(\beta \sum_{a_i=1}^{b-k_i} y_{a_i}\right)\right] + \mathrm{e}^{-\mathrm{i}X_\alpha}\left[1-\tanh\left(\beta \sum_{a_i=1}^{b-k_i} y_{a_i}\right)\right]\right\}$$
$$= \left(\int_0^{2\pi} \frac{\mathrm{d}X}{2\pi} \prod_{i=1}^{N} \left\{\mathrm{e}^{\mathrm{i}X}\left[1+\tanh\left(\beta \sum_{a_i=1}^{b-k_i} y_{a_i}\right)\right] + \mathrm{e}^{-\mathrm{i}X}\left[1-\tanh\left(\beta \sum_{a_i=1}^{b-k_i} y_{a_i}\right)\right]\right\}\right)^m. \tag{2.37}$$

We now proceed to take the limit $m \to 0$. By using:

$$\lim_{m\to 0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \pi(h)\pi(h')\,\mathrm{d}h\,\mathrm{d}h' \frac{\cosh^m(\beta J)}{\cosh^m(\beta z)} \mathrm{e}^{\beta z m} - \cosh^m(\beta J)$$
$$\approx -m \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \pi(h)\pi(h')\,\mathrm{d}h\,\mathrm{d}h'[\ln\cosh(\beta z) - \beta z] \tag{2.38}$$

and after much tedious algebra, we get

$$\lim_{m\to 0} \langle Z^m \rangle_{\mathrm{av}} \approx \frac{1}{g} \exp\left\{-N\left[\frac{b}{2} r^2 - \ln\left(\frac{(\sqrt{Nb})^b}{b!}\right) - b \ln r\right] - NX\right\} \tag{2.39}$$

$$X \equiv \left(\frac{b}{2} mr^2 - bm\right) \ln\cosh(\beta J) + \frac{bm}{r} \int_{-\infty}^{\infty} \pi(h)\,\mathrm{d}h \ln\cosh(\beta y)$$
$$- \frac{bm}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \pi(h)\pi(h')\,\mathrm{d}h\,\mathrm{d}h'[\ln\cosh(\beta z) - \beta z]$$
$$- \frac{m}{r^b} \sum_{k=0}^{b} \binom{b}{k} (r-1)^k \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{a=1}^{b-k} [\pi(h_a)\,\mathrm{d}h_a] \ln\cosh\left(\beta \sum_{a=1}^{b-k} y_a\right)$$
$$- \frac{m}{Nr^{Nb}} \sum \prod_{i=1}^{N} \left[\binom{b}{k_i} (r-1)^{k_i} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{a_i=1}^{b-k_i} [\pi(h_{a_i})\,\mathrm{d}h_{a_i}]\right] \ln[\mathrm{int}(1)] \tag{2.40}$$

$$\mathrm{int}(1) \equiv \int_0^{2\pi} \frac{\mathrm{d}X}{2\pi} \prod_{i=1}^{N} \left\{\mathrm{e}^{\mathrm{i}X}\left[1+\tanh\left(\beta \sum_{a_i=1}^{b-k_i} y_{a_i}\right)\right] + \mathrm{e}^{-\mathrm{i}X}\left[1-\tanh\left(\beta \sum_{a_i=1}^{b-k_i} y_{a_i}\right)\right]\right\}. \tag{2.41}$$

From equation (2.18), we see that the saddle-point solution for r must be $r=1$, so that the limit $m\to 0$ can be consistently carried out. We finally arrive at:

$$\lim_{N\to\infty}\frac{\langle\ln Z\rangle_{\text{av}}}{N}\approx\frac{b}{2}\ln\cosh(\beta J)+\frac{b}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\pi(h)\pi(h')\,\mathrm{d}h\,\mathrm{d}h'[\ln\cosh(\beta z)-\beta z]$$

$$-b\int_{-\infty}^{\infty}\pi(h)\,\mathrm{d}h\ln\cosh(\beta y)$$

$$+\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\prod_{a=1}^{b}[\pi(h_a)\,\mathrm{d}h_a]\ln\cosh\left(\beta\sum_{a=1}^{b}y_a\right)$$

$$+\frac{1}{N}\prod_{i=1}^{N}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\prod_{a_i=1}^{b}[\pi(h_{a_i})\,\mathrm{d}h_{a_i}]\ln(\text{int}) \qquad (2.42)$$

$$\text{int}\equiv\int_0^{2\pi}\frac{\mathrm{d}X}{2\pi}\prod_{i=1}^{N}\left\{\mathrm{e}^{\mathrm{i}X}\left[1+\tanh\left(\beta\sum_{a_i=1}^{b}y_{a_i}\right)\right]+\mathrm{e}^{-\mathrm{i}X}\left[1-\tanh\left(\beta\sum_{a_i=1}^{b}y_{a_i}\right)\right]\right\}. \qquad (2.43)$$

Equations (2.42) and (2.43) are valid for any T. However, the complexity of the last term makes it impossible to carry out the numerical calculation at finite T. Fortunately, the algebra simplifies considerably at $T=0$. On physical grounds, $\pi(h)$ might have, at $T=0$, the following form:

$$\pi(h)=\pi_0\delta(h)+\sum_{l=1}^{b}\pi_l^+\delta(h-lJ)+\sum_{l=1}^{b}\pi_l^-\delta(h+lJ). \qquad (2.44)$$

In the appendix, we shall check the self-consistency of the above ansatz. Let

$$Q=\sum_{l=1}^{b}(\pi_l^++\pi_l^-)\qquad\text{and}\qquad R=\sum_{l=1}^{b}(\pi_l^+-\pi_l^-).$$

By using:

$$\lim_{\beta\to\infty}\tanh^{-1}[\tanh(\beta J)\tanh(\beta h)]\equiv\lim_{\beta\to\infty}\beta y=\begin{cases}\beta J & \text{if } h\geqslant J\\ \beta h & \text{if } 0\leqslant h\leqslant J\end{cases} \qquad (2.45)$$

we finally arrive at:

$$\lim_{\beta\to\infty}\lim_{N\to\infty}\frac{\langle\ln Z\rangle_{\text{av}}}{N\beta J}=\frac{b}{2}(Q^2-R^2)+b(\tfrac{1}{2}-Q)$$

$$+\sum_{u=0}^{b}\sum_{v=0}^{u}\frac{b!(1-Q)^v[\frac{1}{2}(Q+R)]^{u-v}[\frac{1}{2}(Q-R)]^{b-u}}{v!(u-v)!(b-u)!}|2u-v-b|$$

$$+\frac{1}{N\beta J}\sum\left(\prod_{i=1}^{N}\frac{b!(1-Q)^{v_i}[\frac{1}{2}(Q+R)]^{u_i-v_i}[\frac{1}{2}(Q-R)]^{b-u_i}}{(v_i)!(u_i-v_i)!(b-u_i)!}\right)\ln(G) \qquad (2.46)$$

$$G=\int_0^{2\pi}\frac{\mathrm{d}X}{2\pi}\prod_{i=1}^{N}\{\mathrm{e}^{\mathrm{i}X}[1+\tanh(2u_i-v_i-b)\beta J]+\mathrm{e}^{-\mathrm{i}X}[1-\tanh(2u_i-v_i-b)\beta J]\} \qquad (2.47)$$

where Σ denotes the sum over all possible combinations of $\{u_i, v_i;\ i=1,2,\ldots,N\}$ for a given fixed $b \in \{1,2,\ldots\}$. Let us now concentrate on equation (2.47). There are total of $N!/(N/2)!(N/2)!$ terms that survive the integration over X. Each one of them is of the following form:

$$\prod_{i=1}^{N/2}\{1+\tanh[2\sigma(u_i)-\sigma(v_i)-b]\beta J\}\prod_{i=N/2+1}^{N}\{1-\tanh[2\sigma(u_i)-\sigma(v_i)-b]\beta J\} \tag{2.48}$$

where σ is a permutation on the set $\{u_i, v_i;\ i=1,2,\ldots,N\}$.

Using the fact that:

$$\lim_{X\to\infty}\ln(1-\tanh X)\simeq\ln 2-2X+\mathrm{O}(\mathrm{e}^{-2x}) \tag{2.49}$$

we observe the following. In the product

$$\prod_{i=1}^{N}\{\mathrm{e}^{\mathrm{i}X}[1+\tanh(2u_i-v_i-b)\beta J]+\mathrm{e}^{-\mathrm{i}X}[1-\tanh(2u_i-v_i-b)\beta J]\} \tag{2.50}$$

if there are less than $N/2+1$ terms with $2u_i-v_i-b>0$ and less than $N/2+1$ terms with $2u_i-v_i-b<0$, then the dominant term, surviving the integration and in the limit $\beta\to\infty$, is 2^{N-M} where M is the number of terms with $2u_i-v_i-b=0$. In this case, there is no contribution to the ground-state energy.

In order to understand more about the last term in equation (2.46), let us look at the collection of sets $\{u_i, v_i;\ i=1,2,\ldots,N\}$, with exactly $N/2+1$ number of terms satisfying $2u_i-v_i-b>0$. First, we introduce the following notation:

$$\mathrm{frac}(n)=\sum_{\substack{u=0\\2u-v-b=n}}^{b}\sum_{v=0}^{u}\frac{b!(1-Q)^v[\frac{1}{2}(Q+R)]^{u-v}[\frac{1}{2}(Q-R)]^{b-u}}{v!(u-v)!(b-u)!}. \tag{2.51}$$

By its very definition, frac(n) is the fraction of spins feeling nJ local field at $T=0$. Then, we see that

$$\frac{N!}{(N/2+1)!(N/2-1)!}\left(\sum_{n=1}^{b}\mathrm{frac}(n)\right)^{N/2+1}\left(\sum_{n=-b}^{0}\mathrm{frac}(n)\right)^{N/2-1} \tag{2.52}$$

generates all possible sets in this collection. If a particular set contains at least one term with $2u-v-b=1$, it will contribute $-2\beta J$, in the limit $\beta\to\infty$. If a particular set contains no terms with $2u-v-b=1$, but at least one term with $2u-v-b=2$, it will contribute $-4\beta J$ in the ground state.

We can apply similar considerations to any collection of sets with exactly $N/2+l$, $l=1,2,\ldots,N/2$ number of terms satisfying either $2u_i-v_i-b>0$ or $2u_i-v_i-b<0$. Then by collecting only terms that are extensive, we get

$$\begin{aligned}\lim_{\beta\to\infty}\lim_{N\to\infty}\frac{1}{N\beta J}\sum\left(\prod_{i=1}^{N}\frac{b!(1-Q)^{v_i}[\frac{1}{2}(Q+R)]^{u_i-v_i}[\frac{1}{2}(Q-R)]^{b-u_i}}{(v_i)!(u_i-v_i)!(b-u_i)!}\right)\ln(G)\\ =-\frac{2}{N}\sum_{l=1}^{N/2}\frac{lN!}{(N/2+l)!(N/2-l)!}\Bigg[\left(\sum_{n=1}^{b}\mathrm{frac}(n)\right)^{N/2+l}\left(\sum_{n=-b}^{0}\mathrm{frac}(n)\right)^{N/2-l}\\ +\left(\sum_{n=-b}^{-1}\mathrm{frac}(n)\right)^{N/2+l}\left(\sum_{n=0}^{b}\mathrm{frac}(n)\right)^{N/2-l}\Bigg].\end{aligned} \tag{2.53}$$

As $N\to\infty$, the binomial distribution approaches the normal distribution asymptotically.

We finally arrive at:

$$\lim_{N\to\infty} -\frac{E_0}{NJ} = \tfrac{1}{2}b(Q^2-R^2)+b(\tfrac{1}{2}-Q)+\sum_{n=-b}^{b}\mathrm{frac}(n)|n|+B+C \tag{2.54}$$

$$B=\begin{cases}0 & \text{if } \sum_{n=-b}^{0}\mathrm{frac}(n)\geq\tfrac{1}{2}\\ 2\sum_{n=-b}^{0}\mathrm{frac}(n)-1 & \text{if } \sum_{n=-b}^{0}\mathrm{frac}(n)\leq\tfrac{1}{2}\end{cases} \tag{2.55}$$

$$C=\begin{cases}0 & \text{if } \sum_{n=0}^{b}\mathrm{frac}(n)\geq\tfrac{1}{2}\\ 2\sum_{n=0}^{b}\mathrm{frac}(n)-1 & \text{if } \sum_{n=0}^{b}\mathrm{frac}(n)\leq\tfrac{1}{2}\end{cases} \tag{2.56}$$

$$\frac{N_c}{N}=\frac{b}{4}+\frac{E_0}{2NJ}. \tag{2.57}$$

The strict enforcement of the constraint $\Sigma_{i=1}^{N} S_i=0$ introduces two extra terms, namely B and C. The reason that we have two extra terms, instead of one, is due to $R\leftrightarrow -R$ symmetry in the problem. We now have to look for saddle-point solutions of equation (2.54). There are two sets of saddle-point solutions. The first is the spin glass solution. The equation satisfied by Q is:

$$Q=1-\frac{(b-1)!}{2}\sum_{u=0}^{b}\sum_{v=0}^{u}\left(\frac{(b-v)(1-Q)^v(\frac{1}{2}Q)^{b-v-1}}{v!(u-v)!(b-u)!}-\frac{2(1-Q)^{v-1}(\frac{1}{2}Q)^{b-v}}{(v-1)!(u-v)!(b-u)!}\right)|2u-v-b| \tag{2.58}$$

and $R=0$ for all $b\geq 3$. It is easy to verify that, for $b\geq 3$, this solution gives

$$\mathrm{frac}(n)=\mathrm{frac}(-n) \tag{2.59}$$

$$\sum_{n=-b}^{0}\mathrm{frac}(n)=\sum_{n=0}^{b}\mathrm{frac}(n)\geq\tfrac{1}{2}. \tag{2.60}$$

Equation (2.60) approaches $\frac{1}{2}$ only when $b\to\infty$. This is the solution obtained by Mézard and Parisi [7] and by Wong and Sherrington [11].

However, there is another saddle-point solution to equation (2.54). It is a well known fact that when we look for extrema in a closed set, we have to check both the boundary (in this case, among those Q and R satisfying $\Sigma_{n=-b}^{0}\,\mathrm{frac}(n)=\frac{1}{2}$) and the interior (among those Q and R satisfying $\Sigma_{n=-b}^{0}\,\mathrm{frac}(n)<\frac{1}{2}$). When we look for the saddle points within the interior, we discover non-physical solutions. ($Q=R=1$ for all $b=3,4,5$, etc.) Hence we have to look for the saddle points on the boundary. That is, we first demand:

$$\sum_{n=-b}^{0}\mathrm{frac}(n)=\tfrac{1}{2}. \tag{2.61}$$

Then among those Q and R satisfying equation (2.61), we look for the saddle points of:

$$f(Q,R,b)=\tfrac{1}{2}b(Q^2-R^2)+b(\tfrac{1}{2}-Q)+\sum_{n=-b}^{b}\mathrm{frac}(n)|n|. \tag{2.62}$$

The physical picture is also clear. With this solution, we let those spins feeling positive local fields point up. The rest of the spins, feeling zero or negative local fields, must point down in order to satisfy the constraint $\Sigma_{i=1}^{N} S_i = 0$. This solution breaks the symmetry of frac(n)↔frac($-n$) maximally, and thus the frustration is not as severe as in the spin glass solution. Hence it will give a lower cost function than that given by the spin glass solution.

As shown in table 1, R decreases monotonically as a function of b. R approaches 0 only when $b \to \infty$. As $b \to \infty$, we recover the replica symmetric solution of the SK model. For $b = 1, 2, 3$, we have

$$\sum_{n=-1}^{0} \mathrm{frac}(n) = 1 - \tfrac{1}{2}(Q + R) \tag{2.63}$$

$$\sum_{n=-2}^{0} \mathrm{frac}(n) = 1 - Q - R + \tfrac{1}{2}QR + \tfrac{3}{4}Q^2 - \tfrac{1}{4}R^2 \tag{2.64}$$

$$\sum_{n=-3}^{0} \mathrm{frac}(n) = 1 - \tfrac{3}{2}Q - \tfrac{3}{2}R + \tfrac{9}{4}Q^2 - \tfrac{3}{4}R^2 + \tfrac{3}{2}QR - \tfrac{5}{4}Q^3 + \tfrac{1}{4}R^3 - \tfrac{3}{4}Q^2R + \tfrac{3}{4}QR^2 \tag{2.65}$$

$$f(Q, R, 1) = \tfrac{1}{2}(Q^2 - R^2) + \tfrac{1}{2} \tag{2.66}$$

$$f(Q, R, 2) = 1 \tag{2.67}$$

$$f(Q, R, 3) = \tfrac{3}{2} - \tfrac{3}{2}(Q^2 - R^2 - Q^3 + QR^2). \tag{2.68}$$

For $b = 1$, $\Sigma_{n=-1}^{0}$ frac(n) $= \frac{1}{2}$ implies $Q + R = 1$. But then equation (2.66) has no saddle point. For $b = 2$, equation (2.67) is independent of Q and R, and hence the saddle point is undetermined. These are the mathematical statements of the fact that the system has no phase transition. That is, with probability 1, the system has no infinite cluster for $b = 1$ and 2. Hence there is no phase transition in the system and the order parameters do not exist. However, for $b \geqslant 3$, we have non-trivial saddle points for equation (2.62), subject to the constraint in equation (2.61). For b between 3 and 9, the saddle-point solutions are listed in the table.

Table 1. Order parameters and cost per site of the spin glass solution and our new solution, for b between 3 and 9. $R_{SG} = 0$ for all $b \geqslant 3$ in the spin glass solution.

b	3	4	5	6	7	8	9
Q_{SG}	0.6667	0.8000	0.7712	0.8329	0.8175	0.8538	0.8446
cost_{SG}	0.1111	0.2560	0.4043	0.5724	0.7392	0.9189	1.097
Q	0.6820	0.7881	0.7811	0.8256	0.8246	0.8491	0.8499
R	0.1484	0.1255	0.0904	0.0840	0.0657	0.0630	0.0519
cost	0.1057	0.2471	0.3962	0.5621	0.7306	0.9087	1.088

3. Discussion

When we replace the constraint, $\Sigma_{i=1}^{N} S_i = 0$, by an exponential penalty term, namely, $\exp[-\lambda(\Sigma_{i=1}^{N} S_i)^2]$ with $\lambda \gg 1$, in the calculation of the canonical partition function, we inevitably arrive at the spin glass solution. This solution, even at $T = 0$, has a non-zero fraction of the so-called 'crazy spins' [7, 11], i.e. those spins with zero local field. This huge amount of entropy, associated with the crazy spins, compromises the exponential penalty term, which is introduced to project out the correct solution satisfying $\Sigma_{i=1}^{N} S_i = 0$. Apparently, the optimal solution has less entropy, and is thus harder to find.

When $b \to \infty$, our optimal solution approaches the SK replica symmetric solution. Hence there might be replica symmetry breaking for $b \geqslant 3$. The question of stability is left for future development.

Finally we want to point out that it is possible to study a model of the graph-bipartitioning problem, which will contain as two limiting cases the problem with average, finite valence and that with fixed, finite valence. We can then study how the threshold of non-zero cost function changes from $2 \ln 2$ to 2. It is defined as the following:

$$J_{ij} \in \{0, J\} \qquad \text{with} \qquad J_{ii} = 0 \tag{3.1}$$

$$\sum_{i=lc+1}^{(l+1)c} \sum_{j=1}^{N} J_{ij} = cbJ \qquad \text{for} \qquad l = 0, 1, 2, \ldots, \frac{N}{c} - 1 \tag{3.2}$$

where c, N/c, cb are integers and $cb \in \{c, c+1, c+2, \ldots\}$. Then $c = 1$ would correspond to the problem with fixed, finite valence; $c = N$ would correspond to the problem with average, finite valence.

Acknowledgments

The author would like to thank K Y M Wong and D Sherrington for sending him the preprint [11] prior to publication. This work was supported in part by US National Science Foundation grant DMR 8518163.

Appendix

In this appendix, we check the self-consistency of the assumptions in equations (2.30), (2.31) and (2.44). From equations (2.28) and (2.29), we first derive the saddle-point equation satisfied by r:

$$r = \lim_{m \to 0} \left(\int_0^{2\pi} \left(\sum_{\alpha=1}^{m} \frac{dX_\alpha}{2\pi} \right) \left[\operatorname*{Tr}_{\{S^\alpha\}} \exp\left(i \sum_{\alpha=1}^{m} X_\alpha S^\alpha \right) G^b \right]^{N-1} \left[\operatorname*{Tr}_{\{S^\alpha\}} \exp\left(i \sum_{\alpha=1}^{m} X_\alpha S^\alpha \right) G^{b-1} \right] \right.$$
$$\left. \times \left\{ \int_0^{2\pi} \left(\sum_{\alpha=1}^{m} \frac{dX_\alpha}{2\pi} \right) \left[\operatorname*{Tr}_{\{S^\alpha\}} \exp\left(i \sum_{\alpha=1}^{m} X_\alpha S^\alpha \right) G^b \right]^N \right\}^{-1} \right) \tag{A1}$$

$$G \equiv A_0 r + \sum_{l=1}^{\infty} A_l Q_l \sum_{\alpha_1 < \ldots < \alpha_l}^{m} S^{\alpha_1} \ldots S^{\alpha_l}. \tag{A2}$$

However, we notice that:

$$1 = \lim_{m \to 0} \left(\int_0^{2\pi} \left(\sum_{\alpha=1}^{m} \frac{dX_\alpha}{2\pi} \right) \left[\operatorname*{Tr}_{\{S^\alpha\}} \exp\left(i \sum_{\alpha=1}^{m} X_\alpha S^\alpha \right) G^b \right]^N \right.$$
$$\left. \times \left\{ \int_0^{2\pi} \left(\sum_{\alpha=1}^{m} \frac{dX_\alpha}{2\pi} \right) \left[\operatorname*{Tr}_{\{S^\alpha\}} \exp\left(i \sum_{\alpha=1}^{m} X_\alpha S^\alpha \right) G^b \right]^N \right\}^{-1} \right)$$
$$= \lim_{m \to 0} A_0 r^2 + \lim_{m \to 0} \left(\int_0^{2\pi} \left(\sum_{\alpha=1}^{m} \frac{dX_\alpha}{2\pi} \right) \right.$$
$$\times \left[\operatorname*{Tr}_{\{S^\alpha\}} \exp\left(i \sum_{\alpha=1}^{m} X_\alpha S^\alpha \right) G^b \right]^{N-1} \left[\operatorname*{Tr}_{\{S^\alpha\}} \exp\left(i \sum_{\alpha=1}^{m} X_\alpha S^\alpha \right) G^{b-1} (G - A_0 r) \right]$$
$$\left. \times \left\{ \int_0^{2\pi} \left(\sum_{\alpha=1}^{m} \frac{dX_\alpha}{2\pi} \right) \left[\operatorname*{Tr}_{\{S^\alpha\}} \exp\left(i \sum_{\alpha=1}^{m} X_\alpha S^\alpha \right) G^b \right]^N \right\}^{-1} \right). \tag{A3}$$

The last term in equation (A3) is proportional to m. Hence $1 = r^2$ and $r = 1$ is one of the saddle points.

The saddle-point equation satisfied by Q_l is:

$$Q_l = \lim_{m\to 0}\left(\int_0^{2\pi}\left(\sum_{\alpha=1}^{m}\frac{dX_\alpha}{2\pi}\right)\left[\operatorname*{Tr}_{\{S^\alpha\}}\exp\left(i\sum_{\alpha=1}^{m}X_\alpha S^\alpha\right)G^b\right]^{N-1}\right.$$
$$\times\left[\operatorname*{Tr}_{\{S^\alpha\}}S^{\beta_1}\ldots S^{\beta_l}\exp\left(i\sum_{\alpha=1}^{m}X_\alpha S^\alpha\right)G^{b-1}\right]$$
$$\left.\times\left\{\int_0^{2\pi}\left(\sum_{\alpha=1}^{m}\frac{dX_\alpha}{2\pi}\right)\left[\operatorname*{Tr}_{\{S^\alpha\}}\exp\left(i\sum_{\alpha=1}^{m}X_\alpha S^\alpha\right)G^b\right]^N\right\}^{-1}\right). \tag{A4}$$

We now check the self-consistency of equations (2.30) and (2.31). Substituting equation (2.30) into the right-hand side of (A4) and following the derivations of equations (2.36) and (2.37), it is easy to show that:

$$Q_l = \frac{\Sigma}{r^{Nb}}\prod_{i=1}^{N-1}\left[\binom{b}{k_i}(r-1)^{k_i}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\prod_{a_i=1}^{b-k_i}[\pi(h_{a_i})\,dh_{a_i}]\right]$$
$$\times\left[\sum_{k_N=0}^{b-1}\binom{b-1}{k_N}(r-1)^{k_N}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\prod_{a_N=1}^{b-1-k_N}[\pi(h_{a_N})\,dh_{a_N}]\right]\times(\text{ratio})^l \tag{A5}$$

$$\text{ratio} = \int_0^{2\pi}\frac{dX}{2\pi}\prod_{i=1}^{N-1}(\text{sum}_+(i))(\text{sum}_-(N))\left(\int_0^{2\pi}\frac{dX}{2\pi}\prod_{i=1}^{N-1}(\text{sum}_+(i))(\text{sum}_+(N))\right)^{-1} \tag{A6}$$

$$\text{sum}_+(i) = e^{iX}\left[1+\tanh\left(\beta\sum_{a_i=1}^{b-k_i}y_{a_i}\right)\right] + e^{-iX}\left[1-\tanh\left(\beta\sum_{a_i=1}^{b-k_i}y_{a_i}\right)\right] \tag{A7}$$

$$\text{sum}_-(N) = e^{iX}\left[1+\tanh\left(\beta\sum_{a_N=1}^{b-1-k_N}y_{a_N}\right)\right] - e^{-iX}\left[1-\tanh\left(\beta\sum_{a_N=1}^{b-1-k_N}y_{a_N}\right)\right] \tag{A8}$$

where Σ denotes the sum over all possible sets of $\{k_i;\, i = 1, 2, \ldots, N-1\}$ with $k_i \in \{0, 1, 2, \ldots, b\}$. $\text{sum}_+(N)$ has the same form as that given in equation (A7), except that b is replaced by $b-1$ and i is replaced by N. We see that in the denominator of ratio, there are $N!/(N/2)!(N/2)!$ terms that survive the integration over X. All of them are non-negative. In the numerator, we have the same number of terms surviving the integration over X. However, they are mixed with + and − signs. Hence we must have

$$-1 \leqslant \text{ratio} \leqslant 1 \quad \text{for all } \beta. \tag{A9}$$

We thus can formally define

$$\tanh(\beta h) \equiv \text{ratio}. \tag{A10}$$

For any given β, equation (A10) gives a one-to-one correspondence between h and ratio. We also see that

$$\frac{\Sigma}{r^{Nb}}\prod_{i=1}^{N-1}\left[\binom{b}{k_i}(r-1)^{k_i}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\prod_{a_i=1}^{b-k_i}[\pi(h_{a_i})\,dh_{a_i}]\right]$$
$$\times\left[\sum_{k_N=0}^{b-1}\binom{b-1}{k_N}(r-1)^{k_N}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\prod_{a_N=1}^{b-1-k_N}[\pi(h_{a_N})\,dh_{a_N}]\right] = \frac{1}{r} = 1. \tag{A11}$$

Hence it is self-consistent to define

$$Q_l \equiv \int_{-\infty}^{\infty} \pi(h) \tanh^l(\beta h)\, \mathrm{d}h$$

with

$$\int_{-\infty}^{\infty} \pi(h)\, \mathrm{d}h = 1$$

and $\pi(h)$ non-negative for all h.

Next we check the self-consistency of the ansatz in equation (2.44). Substituting equation (2.44) into equation (A5) and using the fact that $r = 1$, we get

$$\lim_{\beta\to\infty} Q_l \equiv \prod_{i=1}^{N-1} \left(\sum_{u_i=0}^{b} \sum_{v_i=0}^{u_i} \frac{b!(1-Q)^{v_i}[\frac{1}{2}(Q+R)]^{u_i-v_i}[\frac{1}{2}(Q-R)]^{b-u_i}}{(v_i)!(u_i-v_i)!(b-u_i)!} \right)$$
$$\times \left(\sum_{u_N=0}^{b-1} \sum_{v_N=0}^{u_N} \frac{(b-1)!(1-Q)^{v_N}[\frac{1}{2}(Q+R)]^{u_N-v_N}[\frac{1}{2}(Q-R)]^{b-1-u_N}}{(v_N)!(u_N-v_N)!(b-1-u_N)!} \right)$$
$$\times \lim_{\beta\to\infty} \text{ratio} \tag{A12}$$

$$\lim_{\beta\to\infty} \text{sum}_{\pm}(i) = \mathrm{e}^{\mathrm{i}X}[1+\tanh(\beta n_i J)] \pm \mathrm{e}^{-\mathrm{i}X}[1-\tanh(\beta n_i J)] \tag{A13}$$

where $n_i = 2u_i - v_i - b$ for $i = 1, 2, \ldots, N-1$ and $u_N = 2u_N - v_N - b + 1$. Since

$$\beta h \equiv \tanh^{-1}(\text{ratio}) = \tfrac{1}{2}\ln(1+\text{ratio}) - \tfrac{1}{2}\ln(1-\text{ratio}) \tag{A14}$$

and

$$\lim_{\beta\to\infty} \ln[1-\tanh(x)] \simeq \ln 2 - 2x + \mathrm{O}(\mathrm{e}^{-2x}) \tag{A15}$$

we see that

$$\lim_{\beta\to\infty} \text{ratio} = 0 \;\Rightarrow\; h = 0 \tag{A16}$$

$$\lim_{\beta\to\infty} \text{ratio} = 1 \;\Rightarrow\; h \geqslant J \tag{A17}$$

$$\lim_{\beta\to\infty} \text{ratio} = -1 \;\Rightarrow\; h \leqslant -J \tag{A18}$$

and h must occur at lJ with $l \in \{0, 1, 2, \ldots, b\}$. The reason is that we have $(1 \pm \tanh \beta nJ)$ with n integer in the expression of $\lim_{\beta\to\infty}$ ratio. This verifies the self-consistency of the assumption that, at $T = 0$, the auxiliary field distribution function is a sum of series of δ functions at integer multiples of J.

References

[1] Edwards S F and Anderson P W 1975 *J. Phys. F: Met. Phys.* **5** 965
[2] Fu Y and Anderson P W 1986 *J. Phys. A: Math. Gen.* **19** 1605
[3] Mézard M and Parisi G 1985 *J. Physique Lett.* **46** L771
[4] Baskaran G, Fu Y and Anderson P W 1986 *J. Stat. Phys.* **45** 1
[5] Orland H 1985 *J. Physique Lett.* **46** L763
[6] Liao W 1987 *Phys. Rev. Lett.* **59** 1625

[7] Mézard M and Parisi G 1987 *Europhys. Lett.* **3** 1067
[8] Kanter I and Sompolinsky H 1987 *Phys. Rev. Lett.* **58** 164
[9] Liao W 1987 *J. Phys. A: Math. Gen.* **20** L695
[10] Erdos P and Renyi A 1973 *The Art of Counting* ed J Spencer (Cambridge, MA: MIT)
[11] Wong K Y M and Sherrington D 1987 *J. Phys. A: Math. Gen.* **20** L793
[12] Banavar J R, Sherrington D and Sourlas N 1987 *J. Phys. A: Math. Gen.* **20** L1